

Newton-Hooke limit of Beltrami-de Sitter spacetime, principle of Galilei-Hooke's relativity and postulate on Newton-Hooke universal time

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Abstract

Based on the Beltrami-de Sitter spacetime, we present the Newton-Hooke model under the Newton-Hooke contraction of the BdS spacetime with respect to the transformation group, algebra and geometry. It is shown that in Newton-Hooke space-time, there are inertial-type coordinate systems and inertial-type observers, which move along straight lines with uniform velocity. And they are invariant under the Newton-Hooke group. In order to determine uniquely the Newton-Hooke limit, we propose the Galilei-Hooke's relativity principle as well as the postulate on Newton-Hooke universal time. All results are readily extended to the Newton-Hooke model as a contraction of Beltrami-anti-de Sitter spacetime with negative cosmological constant.

Keywords: Newton-Hooke space-time, Galilei-Hooke's relativity principle, universal time postulate, contraction, Beltrami-de Sitter spacetime

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I. INTRODUCTION

It is well known that there are different kinds of kinematics on homogeneous (3+1)-d space-times [1], and all of them can be contracted from Minkowski, de Sitter or anti-de Sitter spacetimes under group contraction [2], respectively¹. Among them, Newton-Hooke (NH) space-times \mathcal{NH}_\pm introduced in [1] are of special interest as a kind of non-relativistic cosmological models in [3–5].

In this paper, we present a model for the NH space-time \mathcal{NH}_+ based on the Beltrami-de Sitter (BdS) spacetime [6–8], denoted by \mathcal{B}_Λ . We find that in \mathcal{NH}_+ there exist a kind of special coordinate systems in which the test particles, which none force acts upon, move at constant coordinate velocities along straight lines. In the coordinates, these particles look like free particles rather than the particles driven by a repulsive force provided by the positive cosmological constant in other coordinates, say, the ones contracted from the *static* dS universe [5]. At the first glance, this property seems very strange. Why in \mathcal{NH}_+ contracted from the dS spacetime with constant curvature there exist such a kind of uniform-velocity motions? The basic reason is, in fact, that there exist a kind of inertial-type

¹ In the present paper, we use *spacetime* or *space-time* to denote a unified manifold of space and time with or without an invariant metric, respectively.

motions in the Beltrami model of the dS spacetime. Such an inertial-type property comes in \mathcal{NH}_+ from the BdS spacetime \mathcal{B}_Λ .

Among various models of dS spacetimes, the Beltrami model is an important one in which dS spacetime is in analog with Minkowski spacetime in the sense that there are inertial-type coordinates, inertial-type motions and they are invariant under fractional linear transformations of a 10-parameter group — dS group $SO(1, 4)$. It is precisely the Beltrami model [9] of a 4-hyperboloid \mathcal{S}_Λ in the 5-d Minkowski spacetime. In \mathcal{B}_Λ , a set of Beltrami coordinate systems covers \mathcal{S}_Λ patch by patch and test particles and light signals move along the timelike and null geodesics, respectively, at *constant* coordinate velocities. Therefore, they look like in free motions in a spacetime without gravity. Thus, the Beltrami coordinates and observers \mathcal{O}_B in these systems may be regarded as of “global inertial-type”. This is why there is such a merit of the NH contraction from BdS spacetime that in \mathcal{NH}_+ there remain inertial-type coordinates and inertial-type observers.

This also shows that the usual algebraic definition of the NH limit cannot determine the contraction of geometry uniquely because of free choice of coordinates if coordinates have no fundamental physical meaning. In particular, for the same kind of coordinates the contraction depends on what dimension of the coordinates might be chosen. In order to determine the unique NH limit, we should supplement certain physical conditions. We argue that the suitable physical conditions should be the counterpart of Galilean relativity principle in \mathcal{NH}_+ , which is named the Galilei-Hooke’s relativity principle, and the postulate on Newton-Hooke universal time t , which satisfies $|t| < \nu^{-1} = R/c$ and the latter is invariant under the Newton-Hooke contraction of $c, R \rightarrow \infty$, i.e. the limit of infinite signal-velocity c and the curvature radius R of the BdS spacetime. The Newton-Hooke universal time is similar to the Newton’s universal time in Newtonian mechanics in the sense that it is separated from space in the metric.

We also find that there is something interesting in \mathcal{NH}_+ , which is closely related to the violation of Euclid’s fifth axiom. In Newtonian mechanics and special relativity, a particle in inertial motion can be stationary in only one inertial frame. If an inertial frame S' has a relative velocity in another one, S , the relative velocity will be constant at every time and every point. In \mathcal{NH}_+ , however, it is possible that a particle in inertial-type motion can be at rest in two different inertial-type frames. In this case the velocity of S' relative to S will be different from point to point.

This paper is organized as follows. In Sec. II, we briefly review BdS spacetime. In Sec. III we study the contraction of transformation group and geometry of BdS spacetime in the NH limit. We also study motion of particles and light signals in the NH limit of BdS spacetime. In Sec. IV, we show why the usual limiting procedure cannot lead to the unique NH limit. We propose the Galilei-Hooke’s relativity principle and the postulate on Newton-Hooke universal time to uniquely determine the contraction of geometry in the NH limit. In Sec. V, we discuss the behaviors of inertial frames in \mathcal{NH}_+ that are different from Newtonian mechanics and special relativity. Geometric diagrams are also given in this section. Finally, we end with a few concluding remarks. In appendix A, we list some results of the connection and curvature of BdS spacetime and their contraction in the NH limit. In appendix B we use the 5-d Minkowski spacetime to show why the NH limit is not unique.

II. THE BELTRAMI-DE SITTER SPACETIME AND INERTIAL-TYPE MOTION

A. The Beltrami- dS Spacetime

We start with a 4-d hyperboloid \mathcal{S}_Λ embedded in a 5-d Minkowski spacetime with $\eta_{AB} = \text{diag}(1, -1, -1, -1, -1)$:

$$\mathcal{S}_\Lambda : \quad \eta_{AB} \xi^A \xi^B = -R^2, \quad (2.1)$$

$$ds^2 = \eta_{AB} d\xi^A d\xi^B, \quad (2.2)$$

where $R^2 := 3\Lambda^{-1}$, $A, B = 0, \dots, 4$, and Λ is the cosmological constant. Clearly, Eqs.(2.1) and (2.2) are invariant under dS group $\mathcal{G}_\Lambda = SO(1, 4)$.

The BdS spacetime \mathcal{B}_Λ is a dS spacetime defined by the following Beltrami coordinates on \mathcal{S}_Λ patch by patch [6]. Clearly, $\mathcal{B}_\Lambda \simeq \mathcal{S}_\Lambda$. For intrinsic geometry of \mathcal{B}_Λ , there are at least eight patches $U_{\pm\alpha} := \{\xi \in \mathcal{S}_\Lambda : \xi^\alpha \gtrless 0\}$, $\alpha = 1, \dots, 4$. In U_4 , for instance, the Beltrami coordinates are

$$x^\mu|_{U_4} = R\xi^\mu/\xi^4, \quad \mu = 0, \dots, 3; \quad (2.3)$$

$$\xi^4 = ((\xi^0)^2 - \sum_{a=1}^3 (\xi^a)^2 + R^2)^{1/2} > 0. \quad (2.4)$$

It is important that the coordinate transformation in each intersection is a fractional linear transformation. For example, let $y^{\mu'}$ ($\mu' = 0, 1, 2, 4$) be the coordinates on U_3 , then, in $U_3 \cap U_4$, the transformation is $x^\mu = R y^\mu / y^4$ ($\mu = 0, 1, 2$), $x^3 = R^2 / y^4$, which is induced by $T_{4,3} = \xi^3 / \xi^4 = x^3 / R = R / y^4 \in \mathcal{G}_\Lambda$.

In each patch, there are condition and Beltrami metric

$$\sigma(x) = \sigma(x, x) := 1 - R^{-2} \eta_{\mu\nu} x^\mu x^\nu > 0, \quad (2.5)$$

$$ds^2 = [\eta_{\mu\nu} \sigma^{-1}(x) + R^{-2} \eta_{\mu\rho} \eta_{\nu\sigma} x^\rho x^\sigma \sigma^{-2}(x)] dx^\mu dx^\nu. \quad (2.6)$$

Under fractional linear transformations of \mathcal{G}_Λ

$$\begin{aligned} x^\mu &\rightarrow \tilde{x}^\mu = \sigma^{1/2}(a) \sigma^{-1}(a, x) (x^\nu - a^\nu) D_\nu^\mu, \\ D_\nu^\mu &= L_\nu^\mu + R^{-2} \eta_{\nu\rho} a^\rho a^\tau (\sigma(a) + \sigma^{1/2}(a))^{-1} L_\tau^\mu, \\ L &:= (L_\nu^\mu)_{\mu, \nu=0, \dots, 3} \in SO(1, 3), \end{aligned} \quad (2.7)$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ in $U_{\pm\alpha}$, condition (2.5) and metric (2.6) are invariant. Note that Eqs.(2.5)-(2.7) are defined on \mathcal{B}_Λ patch by patch. This is, in fact, a cornerstone for the special relativity-type principle. In addition, at the origin of the coordinate system, the metric (2.6) becomes Minkowskian.

The generators of \mathcal{G}_Λ in Beltrami coordinates are expressed as

$$\begin{aligned} \mathbf{P}_\mu &= (\delta_\mu^\rho - R^{-2} x_\mu x^\rho) \partial_\rho, \quad x_\mu := \eta_{\mu\nu} x^\nu, \\ \mathbf{L}_{\mu\nu} &= x_\mu \mathbf{P}_\nu - x_\nu \mathbf{P}_\mu = x_\mu \partial_\nu - x_\nu \partial_\mu \in so(1, 3). \end{aligned} \quad (2.8)$$

Hereafter, we use bold roman letters to denote generators of Lie group. Later we shall use a bold italic letter \boldsymbol{x} to denote the triple (x^1, x^2, x^3) (see, Eq.(2.33), for example). They form an $so(1, 4)$ algebra:

$$\begin{aligned} [\mathbf{P}_\mu, \mathbf{P}_\nu] &= R^{-2} \mathbf{L}_{\mu\nu} \\ [\mathbf{L}_{\mu\nu}, \mathbf{P}_\rho] &= \eta_{\nu\rho} \mathbf{P}_\mu - \eta_{\mu\rho} \mathbf{P}_\nu \\ [\mathbf{L}_{\mu\nu}, \mathbf{L}_{\rho\sigma}] &= \eta_{\nu\rho} \mathbf{L}_{\mu\sigma} - \eta_{\nu\sigma} \mathbf{L}_{\mu\rho} + \eta_{\mu\sigma} \mathbf{L}_{\nu\rho} - \eta_{\mu\rho} \mathbf{L}_{\nu\sigma}, \end{aligned} \quad (2.9)$$

as expected. The Casimir operators of \mathcal{G}_Λ are (see, e.g., [10])

$$\mathbf{C}_1 := \mathbf{P}_\mu \mathbf{P}^\mu - \frac{1}{2R^2} \mathbf{L}_{\mu\nu} \mathbf{L}^{\mu\nu}, \quad \mathbf{C}_2 := \mathbf{S}_\mu \mathbf{S}^\mu - \frac{1}{R^2} \mathbf{W}^2, \quad (2.10)$$

where

$$\mathbf{P}^\mu = \eta^{\mu\nu} \mathbf{P}_\nu, \quad \mathbf{L}^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} \mathbf{L}_{\rho\sigma}, \quad (2.11)$$

$$\mathbf{S}_\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \mathbf{P}^\nu \mathbf{L}^{\lambda\sigma}, \quad \mathbf{S}^\mu = \eta^{\mu\nu} \mathbf{S}_\nu, \quad (2.12)$$

$$\mathbf{W} = \frac{1}{8} \epsilon_{\mu\nu\lambda\sigma} \mathbf{L}^{\mu\nu} \mathbf{L}^{\lambda\sigma}. \quad (2.13)$$

In Eq.(2.11), $\epsilon_{\mu\nu\lambda\sigma}$ is 4-d Levi-Civita symbol in flat spacetime with $\epsilon_{0123} = 1$.

According to the spirit of Einstein's special relativity, one can define simultaneity such that two events A and B are simultaneous if and only if the Beltrami time coordinates x^0 's for the two events coincide,

$$a^0 := x^0(A) = x^0(B) =: b^0. \quad (2.14)$$

It defines the laboratory time in one patch.

The simultaneity also defines a 3+1 decomposition of spacetime

$$ds^2 = N^2(dx^0)^2 - h_{ij} (dx^i + N^i dx^0) (dx^j + N^j dx^0) \quad (2.15)$$

with the lapse function, shift vector, and induced 3-geometry on 3-hypersurface Σ in one coordinate patch

$$\begin{aligned} N &= \{\sigma_\Sigma(x)[1 - (x^0/R)^2]\}^{-1/2}, \\ N^i &= x^0 x^i [R^2 - (x^0)^2]^{-1}, \\ h_{ij} &= \delta_{ij} \sigma_\Sigma^{-1}(x) - [R\sigma_\Sigma(x)]^{-2} \delta_{ik} \delta_{jl} x^k x^l, \end{aligned} \quad (2.16)$$

respectively, where $\sigma_\Sigma(x) = \sigma(x)|_{x^0=\text{const.}} = 1 - (x^0/R)^2 + \delta_{ij} x^i x^j / R^2|_{x^0=\text{const.}}$, δ_{ij} is the Kronecker δ -symbol, $i, j = 1, 2, 3$. In particular, at $x^0 = 0$, $\sigma_\Sigma(x) = 1 + \delta_{ij} x^i x^j / R^2$.

In \mathcal{B}_Λ , the simultaneity can also be defined with respect to the proper time of a clock rested at the spatial origin of the coordinate system [6]:

$$T_\Lambda = R \sinh^{-1} \left(\frac{ct}{R\sigma^{1/2}(x)} \right). \quad (2.17)$$

If this proper time is chosen as the time coordinate, the metric (2.6) becomes the Robertson-Walker-like one as follows [6]:

$$ds^2 = c^2 dT^2 - \cosh^2 \left(\frac{cT}{R} \right) dl_{\Sigma_T}^2. \quad (2.18)$$

where

$$\begin{aligned} dl_{\Sigma_T}^2 &:= \sigma_{\Sigma_T}^{-2}(x) \left(\sigma_{\Sigma_T}(x) \delta_{ij} - \frac{1}{R^2} \delta_{ik} \delta_{jl} x^k x^l \right) dx^i dx^j, \\ \sigma_{\Sigma_T}(x, x) &:= 1 + \frac{1}{R^2} \delta_{ij} x^i x^j > 0. \end{aligned} \quad (2.19)$$

The metric Eq.(2.18) is closely linked with the cosmological principle.

The two kinds of simultaneity Eqs.(2.14) and (2.17) indicate that in \mathcal{B}_Λ there is a relation between the special relativity-type principle, which could be introduced in the following subsection, and the cosmological principle.

B. The Inertial-type Motion and Beltrami-de Sitter Relativity Principle

It is important to see the advantage of Beltrami coordinates — both test free massive particles and light signals in \mathcal{B}_Λ are moving along timelike straight world lines and the null ones, respectively. In terms of straight lines, we refer to those curves with equations in a linear form. This indicates that both their motions and the Beltrami coordinate systems are of the inertial-type. Furthermore, these properties are invariant under the fractional linear transformations (2.7).

In \mathcal{B}_Λ geodesics, which satisfy

$$\frac{d^2 x^\rho}{ds^2} + \Gamma_{\mu\nu}{}^\rho \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \quad (2.20)$$

with

$$\Gamma_{\mu\nu}{}^\rho = \frac{1}{R^2 \sigma(x)} (x_\mu \delta_\nu^\rho + x_\nu \delta_\mu^\rho), \quad (2.21)$$

are Lobachevski-like straight world lines [6–9]. In particular, timelike geodesics have the form

$$x^\mu(w) = c^\mu w + b^\mu \quad (2.22)$$

with the initial condition

$$x^\mu|_{s=0} = b^\mu, \quad \left. \frac{dx^\mu}{ds} \right|_{s=0} = c^\mu$$

and the constraint

$$g_{\mu\nu}(b) c^\mu c^\nu = 1,$$

where $w = w(s)$ is a new parameter such that

$$w(s) = \frac{R \sinh \frac{s}{R}}{\frac{\eta_{\mu\nu} c^\mu b^\nu}{R \sigma(b)} \sinh \frac{s}{R} + \cosh \frac{s}{R}}. \quad (2.23)$$

Null geodesics have the form

$$x^\mu = c^\mu w + b^\mu, \quad (2.24)$$

with the initial condition

$$x^\mu|_{\lambda=0} = b^\mu, \quad \left. \frac{dx^\mu}{d\lambda} \right|_{\lambda=0} = c^\mu,$$

and the constraint

$$g_{\mu\nu}(b) c^\mu c^\nu = 0,$$

where λ is an affine parameter and $w = w(\lambda)$ is a new parameter such that

$$w(\lambda) = \frac{\lambda}{1 + \lambda/\lambda_0} \quad (2.25)$$

with

$$\lambda_0 = \sqrt{\frac{R^2 \sigma(b)}{|\eta_{\mu\nu} c^\mu c^\nu|}}.$$

Namely, free particles and light signals move along straight lines with a uniform component velocity

$$\frac{dx^i}{dt} = v^i, \quad \frac{d^2x^i}{dt^2} = 0, \quad i = 1, 2, 3, \quad (2.26)$$

which implies that the Beltrami coordinate system is of physical meaning as inertial-type.

Furthermore, for a free particle 4-momentum and 4-angular momentum, p^μ and $L^{\mu\nu}$, defined by

$$p^\mu := \frac{m_{\Lambda 0}}{\sigma(x)} \frac{dx^\mu}{ds}, \quad (2.27)$$

$$L^{\mu\nu} := x^\mu p^\nu - x^\nu p^\mu, \quad (2.28)$$

are conserved along a geodesic, i.e.

$$\frac{dp^\mu}{ds} = 0, \quad (2.29)$$

$$\frac{dL^{\mu\nu}}{ds} = 0. \quad (2.30)$$

They constitute a 5-d angular momentum

$$\mathcal{L}^{AB} := m_{\Lambda 0} (\xi^A \frac{d\xi^B}{ds} - \xi^B \frac{d\xi^A}{ds}) \quad (2.31)$$

for a free particle in \mathcal{S}_Λ in such a way that $\mathcal{L}^{\mu\nu} = L^{\mu\nu}$, $\mathcal{L}^{4\mu} = R p^\mu$, and it is conserved along the geodesic,

$$\frac{d\mathcal{L}^{AB}}{ds} = 0. \quad (2.32)$$

The Einstein's famous formula can be generalized in \mathcal{B}_Λ :

$$-\frac{1}{2R^2} \mathcal{L}^{AB} \mathcal{L}_{AB} = E^2 - \mathbf{P}^2 c^2 + \frac{c^4}{R^2} \mathbf{K}^2 - \frac{c^2}{R^2} \mathbf{J}^2 = m_{\Lambda 0}^2 c^4, \quad (2.33)$$

$$\frac{E}{c} = p^0, \quad P^i = p^i, \quad K^i = \frac{1}{c} L^{0i}, \quad J^i = \frac{1}{2} \epsilon^{ijk} L^{jk}, \quad (2.34)$$

where $\mathcal{L}_{AB} = \eta_{AC} \eta_{BD} \mathcal{L}^{CD}$ and ϵ^{ijk} is totally anti-symmetric with $\epsilon^{123} = 1$. As mentioned in the previous subsection, in this paper we use a bold italic letter, \mathbf{P} , say, to denote the triple (P^1, P^2, P^3) . Notations such as $\mathbf{P}^2 = |\mathbf{P}|^2$, $\mathbf{P} \cdot \mathbf{K}$ and $\mathbf{P} \times \mathbf{A}$ are abbreviations of $(P^1)^2 + (P^2)^2 + (P^3)^2$, $P^1 K^1 + P^2 K^2 + P^3 K^3$ and $\epsilon^{ijk} P^j K^k$, respectively. In this sense, sometimes we do not care whether the summed indices are one upper and one lower, as done in Newtonian mechanics. In addition, Eq.(2.33) gives rise the energy of photons,

$$E_\gamma = \sqrt{\mathbf{P}^2 c^2 - \frac{c^4}{R^2} \mathbf{K}^2 + \frac{c^2}{R^2} \mathbf{J}^2}. \quad (2.35)$$

In any case, these offer a consistent way to define the observables for free particles and this kind of definitions differ from any others in $d\mathcal{S}$ space. Of course, these issues significantly indicate that the motion of a free particle in \mathcal{B}_Λ should be of inertial-type in analog to Newton's and Einstein's conception for the inertial motion of a free particle with constant velocity. Consequently, the coordinate systems with Beltrami metric should be linked with inertial-type frames and corresponding observer should be of inertial-type as well.

The deviation of nearby, ‘same-directed’, straight world lines is governed by the geodesic deviation equation

$$\frac{D^2\zeta^\mu}{ds^2} + R^\mu{}_{\nu\lambda\sigma}\zeta^\lambda\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds} = 0, \quad (2.36)$$

where ζ^μ is the deviation vector of nearby geodesics such that

$$g_{\mu\nu}\zeta^\mu\frac{dx^\nu}{ds} = 0 \quad (2.37)$$

on the straight world lines. For simplicity, consider two static particles in a Beltrami coordinate system, the zero-component of the deviation vector of which is identical to zero. Then, Eqs.(2.36) and (2.37) reduce to

$$\frac{d^2\zeta^\mu}{dt^2} - \frac{2}{R^2}\frac{d(\sigma^{-1}\delta_{ij}x^i\zeta^j)}{dt}\frac{dx^\mu}{dt} = 0 \quad (2.38)$$

and

$$\delta_{ij}x^j\zeta^i = 0. \quad (2.39)$$

They give rise to

$$\frac{d^2\zeta^\mu}{dt^2} = 0. \quad (2.40)$$

Notice that for the two static particles $\frac{d\zeta^\mu}{dt}$ vanishes initially. Thus, measured by use of Beltrami coordinates, the deviation of two static particles in the same system will be unchanged in the evolution[8]. The conclusion is not only valid in a vicinity of the spatial origin of a system, but also valid at other places.

As was noted in [6–8], all above results indicate that a principle similar to the principle of the special relativity for the inertial motion with respect to the inertial coordinates in special relativity could be set up in the Beltrami-de Sitter spacetime \mathcal{B}_Λ . It is named the Beltrami-de Sitter relativity principle.

III. THE NEWTON-HOOKE LIMIT OF BELTRAMI- dS SPACETIME AND INERTIAL-TYPE MOTION

A. The Newton-Hooke Limit of Beltrami- dS Spacetime

In order to consider the non-relativistic limit of \mathcal{B}_Λ , we set $x^0 = ct$ and $x'^0 = ct'$, and correspondingly, we assume that $a^0 = ct_a$ and $b^0 = ct_b$ with t_a and t_b being finite under the limit. Naively, this non-relativistic limit may be attained when we simply let $c \rightarrow \infty$, but this limit is not well defined if R remains finite (for algebraic reasons, see, e.g. [5]). In order to obtain the meaningful non-relativistic limit, one has to consider the so-called Newton-Hooke limit, which is defined as

$$c \rightarrow \infty, \quad R \rightarrow \infty, \quad \text{but} \quad \nu = \frac{c}{R} \text{ is a positive, finite constant.} \quad (3.1)$$

ν^{-1} has the same dimension as time. Furthermore, t , x^i and the primed quantities correspondingly are assumed to be finite under the limit. This is a crucial requirement for the Newton-Hooke limit. Otherwise, the limit will not be unique. In Sec.IV, we shall discuss the problem in details.

Under the NH limit,

$$\lim_{c, R \rightarrow \infty} \sigma(x) = 1 - \nu^2 t^2 := \sigma_+(t), \quad \lim_{c, R \rightarrow \infty} \sigma(a, x) = 1 - \nu^2 t_a t := \sigma_+(t_a, t). \quad (3.2)$$

The condition $\sigma(x) > 0$ can be assured by

$$-\frac{1}{\nu} < t < \frac{1}{\nu}. \quad (3.3)$$

It can be shown that the transformations (2.7) become

$$t' = \frac{t - t_a}{\sigma_+(t_a, t)}, \quad x'^i = \frac{\sigma_+^{1/2}(t_a)}{\sigma_+(t_a, t)} O_j^i (x^j - a^j - u^j t), \quad O_j^i \in SO(3), \quad (3.4)$$

which is called a *Newton-Hooke transformation*. The first equation of Eq.(3.4) shows that t is separated from space in the transformation and that the simultaneity for t is absolute. In this sense, we call t the Newton-Hooke universal time. An NH transformation has inverse transformation, reading

$$t = \frac{t' - t'_a}{\sigma_+(t'_a, t')}, \quad x^i = \frac{\sigma_+^{1/2}(t'_a)}{\sigma_+(t'_a, t')} O_j^i (x'^j - a'^j - u'^j t') \quad (3.5)$$

with

$$t'_a = -t_a, \quad a'^i = -O_j^i \frac{a^j + u^j t_a}{\sigma_+^{1/2}(t_a)}, \quad u'^i = -O_j^i \frac{u^j + \nu^2 t_a a^j}{\sigma_+^{1/2}(t_a)}. \quad (3.6)$$

In the inverse transformation and follows, O_j^i are the entries of the inverse matrix of O^i_j , satisfying $O_k^i O_j^k = \delta_j^i$. So the inverse transformation of an NH transformation is still an NH transformation. Let NH_+ be the set of all the NH transformations. Then it follows that NH_+ is a subset of the group of diffeomorphisms. It can be further shown that an NH transformation with parameters t_a , a^i , u^i and the rotation O^i_j followed by another with parameters t'_a , a'^i , u'^i and the rotation O'^i_j is equivalent to an NH transformation with the following parameters,

$$\begin{aligned} t''_a &= \frac{t_a + t'_a}{1 + \nu^2 t_a t'_a}, & a''^i &= a^i + O_j^i \frac{a'^j - t_a u'^j}{\sigma_+^{1/2}(t_a)}, \\ u''^i &= u^i + O_j^i \frac{u'^j - \nu^2 t_a a'^j}{\sigma_+^{1/2}(t_a)}, & O''^i_j &= O'^i_k O^k_j. \end{aligned} \quad (3.7)$$

In short, the composition of two NH transformations are still an NH transformation. Since NH_+ , the subset of the group of diffeomorphisms, is closed under the inverse operation and multiplication, it follows that NH_+ is a subgroup. We call NH_+ the Newton-Hooke group. It is a fractional linear realization of a Lie group, having 10 free parameters, t_a, a^i, u^i , and three parameters in O^i_j .

Note that the transformation for time coordinate is independent of space coordinates, and that the time transformation is still fractional linear while the transformation for space coordinates are linear among themselves. In other words, if two events have the same value for coordinate t , they have the same value for t' . Thus the simultaneity defined by the ‘time’ coordinate (either t or t') is absolute under the action of the NH group. Such a space-time is called the Newton-Hooke space-time, denoted by \mathcal{NH}_+ as before.

From the transformation (3.4), the ‘time-translation’ generator with respect to t_a reads

$$\mathbf{H} = \sigma_+(t)\partial_t - \nu^2 t x^i \partial_i. \quad (3.8)$$

Similarly, the ‘space-translation’ generators with respect to a^i and boost generators with respect to u^i can be easily obtained as

$$\mathbf{P}_i = \partial_i, \quad \mathbf{K}_i = t\partial_i, \quad (3.9)$$

respectively. The space-rotation generators are the generators of $SO(3)$ as usual. From the above explicit forms of generators, we can readily write down the following Lie algebra:

$$\begin{aligned} [\mathbf{H}, \mathbf{P}_i] &= \nu^2 \mathbf{K}_i, & [\mathbf{H}, \mathbf{K}_i] &= \mathbf{P}_i, \\ [\mathbf{J}_i, \mathbf{P}_j] &= \epsilon_{ijk} \mathbf{P}_k, & [\mathbf{J}_i, \mathbf{K}_j] &= \epsilon_{ijk} \mathbf{K}_k, & [\mathbf{J}_i, \mathbf{J}_j] &= \epsilon_{ijk} \mathbf{J}_k \end{aligned} \quad (3.10)$$

and other Lie brackets vanish,

where ϵ_{ijk} is totally anti-symmetric with $\epsilon_{123} = 1$. The Lie algebra may also be reached by taking the NH contraction of the Lie algebra (2.9) on BdS noting

$$\mathbf{H} = c\mathbf{P}_0, \quad \mathbf{K}_i = \frac{1}{c}\mathbf{L}_{0i}, \quad \mathbf{J}_i = \frac{1}{2}\epsilon_{ijk}\mathbf{L}_{jk}, \quad (3.11)$$

and it is exactly the same as \mathfrak{n}_{10}^+ in Eq.(13) of [5] as long as setting $\nu^2 = \tau^{-2}$. The first Casimir operator is

$$\tilde{\mathbf{C}}_1 = \mathbf{P}_i \mathbf{P}_i - \nu^2 \mathbf{K}_i \mathbf{K}_i, \quad (3.12)$$

which can also be obtained from the contraction of $-\mathbf{C}_1/c^2$ from Eq.(2.10). The second Casimir operator tends to zero as c (and R) $\rightarrow \infty$. If the NH quantum mechanics and other physical aspects are concerned, we must consider the central extension of \mathfrak{n}_{10}^+ and the corresponding Casimir operators. We will examine these aspects in Ref. [11].

Obviously, the NH limit will not alter the definition of simultaneity, Eq.(2.14). Under the NH limit, Eq.(2.16) turns out to be

$$\begin{aligned} N_+ &= \sigma_+^{-1}(t), \\ cN_+^i &= \nu^2 t x^i \sigma_+^{-1}(t), \\ h_{+ij} &= \sigma_+^{-1}(t) \delta_{ij}, \end{aligned} \quad (3.13)$$

and the invariant Beltrami metric (2.6) now splits to two metrics for time and space, respectively. They are

$$d\tau^2 = \frac{1}{\sigma_+^2(t)} dt^2 = \frac{dt^2}{(1 - \nu^2 t^2)^2}, \quad (3.14)$$

and in a hypersurface of simultaneity

$$dl^2 = \sigma_+^{-1}(t) dl_0^2, \quad dl_0^2 := \delta_{ij} dx^i dx^j. \quad (3.15)$$

Note that under the NH limit the second definition of simultaneity with respect to the proper time (2.17) of a clock rested at the spatial origin of the coordinate system in \mathcal{B}_Λ now coincides with the first one. This implies that $d\tau^2$ on the whole \mathcal{NH}_+ and dl^2 on each hypersurface of simultaneity

are defined intrinsically. Namely, they are invariant under the NH transformations. We can examine the fact, using Eq.(3.4) and formulae

$$\sigma_+(t') = \frac{\sigma_+(t_a)\sigma_+(t)}{\sigma_+^2(t_a, t)}, \quad (3.16)$$

$$dt' = \frac{\sigma_+(t_a)}{\sigma_+^2(t_a, t)} dt, \quad (3.17)$$

and in the hypersurface of simultaneity ($dt = dt' = 0$)

$$dx'^i = \frac{\sigma_+^{1/2}(t_a)}{\sigma_+(t_a, t)} dx^i. \quad (3.18)$$

The fact can also be checked directly by calculating the Lie derivatives of $d\tau^2$ and dl^2 with respect to all the generators.

Under the NH limit, the Robertson-Walker-like metric (2.18) in \mathcal{B}_Λ also splits to two parts. The detailed discussion is left in the next section.

B. Inertial-type Motion of Test Particles and Signals in \mathcal{NH}_+

It should be noted that the space-time in the NH limit is not a 4-d metric spacetime because there is no invariant metric on it. In spite of that, a connection still exists as the contraction of the connection on \mathcal{B}_Λ . Let the coefficients of the connection on \mathcal{NH}_+ be denoted by $\Gamma_{\mu\nu}^\rho$, where the indices take values from $t, 1, 2$ and 3 . Then a direct calculation results in all of them, among which only the following are nonzero:

$$\Gamma_{tt}^t = \frac{2\nu^2 t}{1 - \nu^2 t^2}, \quad \Gamma_{tj}^i = \Gamma_{jt}^i = \frac{\nu^2 t}{1 - \nu^2 t^2} \delta_i^j. \quad (3.19)$$

A further discussion on connection and curvature in \mathcal{NH}_+ is made in Appendix A.

Under the NH limit, the geodesic equation (2.20) becomes

$$\frac{d^2 t}{d\tau^2} + \Gamma_{tt}^t \frac{dt}{d\tau} \frac{dt}{d\tau} = 0, \quad (3.20)$$

$$\frac{d^2 x^i}{d\tau^2} + 2\Gamma_{tj}^i \frac{dt}{d\tau} \frac{dx^j}{d\tau} = 0. \quad (3.21)$$

The general solution of Eq.(3.20) is

$$t = \nu^{-1} \tanh(C_1 + C_2 \nu \tau),$$

where C_1 and C_2 are two integral constants. Depending on whether $dt/d\tau$ is zero or not in the initial condition, this can be reduced to

$$t = \begin{cases} \text{const} & \text{if } \left. \frac{dt}{d\tau} \right|_{\tau=0} = 0, \\ \frac{1}{\nu} \tanh \nu \tau & \text{if } \left. \frac{dt}{d\tau} \right|_{\tau=0} \neq 0. \end{cases} \quad (3.22)$$

For convenience, we call a curve in \mathcal{NH}_+ timelike if $dt/d\tau$ is nonzero at every parameter τ along it, and spacelike if it is a curve in the hypersurface $t = \text{const}$. A timelike curve is also called a world line of a particle, as did in relativity.

For a free particle with mass $m_{\Lambda 0}$, the second expression of Eq.(3.22) can be directly obtained from Eq.(3.14) under the initial condition $\tau = 0$ when $t = 0$. Eq.(3.21) becomes

$$\frac{d^2 x^i}{d\tau^2} + 2\nu \tanh(\nu\tau) \frac{dx^i}{d\tau} = 0, \quad (3.23)$$

which can be integrated out:

$$x^i = v^i \frac{\tanh(\nu\tau)}{\nu} + b^i. \quad (3.24)$$

This is the solution with the initial condition

$$t|_{\tau=0} = 0, \quad x^i|_{\tau=0} = b^i, \quad \left. \frac{dt}{d\tau} \right|_{\tau=0} = 1, \quad \left. \frac{dx^i}{d\tau} \right|_{\tau=0} = v^i. \quad (3.25)$$

By use of the second expression of Eq.(3.22), the geodesic appears in an explicit form of a straight world line

$$x^i(t) = v^i t + b^i. \quad (3.26)$$

This property is in analog with the straight line in the Beltrami model of Lobachevski plane [9]. The parameter $t = w/c$ can also be obtained from Eq.(2.23) under the initial condition Eq.(3.25) as long as $\eta_{\mu\nu} c^\mu c^\nu \neq 0$. (The situation corresponding to $\eta_{\mu\nu} c^\mu c^\nu \neq 0$ does not appear in the NH limit.)

In the NH limit, free particles move along straight lines at constant velocities

$$\frac{dx^i}{dt} = v^i; \quad \frac{d^2 x^i}{dt^2} = 0. \quad (3.27)$$

Eq.(3.4) gives rise the velocity addition law

$$v'^i := \frac{dx'^i}{dt'} = \frac{O^i_j}{\sigma_+^{1/2}(t_a)} [\sigma_+(t_a, t) v^j - u^j + \nu^2 t_a (x^j - a^j)]. \quad (3.28)$$

In particular, for free particles the above expression becomes

$$v'^i = \frac{O^i_j}{\sigma_+^{1/2}(t_a)} [v^j - u^j + \nu^2 t_a (b^j - a^j)]. \quad (3.29)$$

It is easy to see that free particles remain in uniform-velocity motions along straight lines under time translation, spatial translation, spatial rotation, and boost transformation.

The non-relativistic energy for a particle can be obtained in standard way. To order $\frac{\mathbf{v}^2}{c^2}$, $\frac{\mathbf{x}^2}{R^2}$, and $\frac{x^i v^j}{Rc}$,

$$\sigma(x) = 1 - \nu^2 t^2 + \frac{\mathbf{x}^2}{R^2}, \quad (3.30)$$

and

$$\frac{dt}{d\tau} \approx \sigma(x) \left[1 + \frac{\mathbf{v}^2}{2c^2} - \frac{1}{2} \left(\frac{\mathbf{x}}{R} - \nu t \frac{\mathbf{v}}{c} \right)^2 \right] \quad (3.31)$$

for a test particle. The energy and 3-momentum of a particle are

$$E = \sigma^{-1}(x)m_{\Lambda 0}c^2\frac{dt}{d\tau} \approx m_{\Lambda 0}c^2 + \frac{1}{2}(1 - \nu^2 t^2)m_{\Lambda 0}\mathbf{v}^2 + m_{\Lambda 0}\nu^2 t \mathbf{x} \cdot \mathbf{v} - \frac{1}{2}m_{\Lambda 0}\nu^2 \mathbf{x}^2, \quad (3.32)$$

and

$$P^i = \sigma^{-1}(x)m_{\Lambda 0}\frac{dx^i}{d\tau} \approx m_{\Lambda 0}v^i, \quad (3.33)$$

respectively. The non-relativistic energy for a particle is then

$$E_{nr} \approx \frac{1}{2m_{\Lambda 0}}\mathbf{P}^2 - \frac{\nu^2}{2m_{\Lambda 0}}(t\mathbf{P} - m_{\Lambda 0}\mathbf{x})^2 = \frac{1}{2m_{\Lambda 0}}(\mathbf{P}^2 - \nu^2 \mathbf{K}^2), \quad (3.34)$$

where the second expression in Eq.(3.34) can be obtained directly from the generalized Einstein formula Eq.(2.33). Therefore, the kinetic energy in the NH limit is still $P^2/(2m_{\Lambda 0})$ and the non-relativistic energy for a particle is equal to the sum of its kinetic energy and the (negative) energy contributed from boost. In particular, for a free particle, the non-relativistic energy reduces to

$$E_{nr} \approx \frac{1}{2m_{\Lambda 0}}\mathbf{P}^2 - \frac{1}{2}m_{\Lambda 0}\nu^2 \mathbf{b}^2, \quad (3.35)$$

where b^i is the initial position of the free particle in Eq.(3.26).

As in Newtonian mechanics, time in Beltrami coordinate system is absolute in the NH limit. One only needs to consider the geodesic deviation under condition $\zeta^0 = c(t - t_0) = 0$. Hence, given a timelike geodesic γ in \mathcal{B}_Λ , we can construct a congruence of geodesics $t = \nu^{-1} \tanh \nu \tau$, $x^i = v^i(u)t + b^i(u)$. The deviation is given by

$$\zeta^i(t) = \left. \frac{dv^i(u)}{du} \right|_{u=0} t + \left. \frac{db^i(u)}{du} \right|_{u=0} \quad (3.36)$$

and satisfies

$$\frac{d^2 \zeta^i}{dt^2} = 0 \quad (3.37)$$

obviously. Eq.(3.37) can also be obtained from the contraction of Eq.(2.36). It is invariant from one inertial-type frame to another inertial-type frame.

Similarly, a light signal moves globally along a null geodesic in \mathcal{B}_Λ . The null geodesic equations formally still have the forms of Eqs.(3.20) and (3.21), but now

$$\frac{dt}{d\lambda} = 0, \quad (3.38)$$

where λ is an affine parameter. Therefore, a null geodesic in \mathcal{B}_Λ becomes a spacelike geodesic in \mathcal{NH}_+ , which can be integrated as a straight line

$$x^i = c^i \lambda + b^i \quad (3.39)$$

from Eq.(3.21) under the initial condition

$$t|_{\lambda=0} = 0, \quad x^i|_{\lambda=0} = b^i, \quad \left. \frac{dt}{d\lambda} \right|_{\lambda=0} = 0, \quad \left. \frac{dx^i}{d\lambda} \right|_{\lambda=0} = c^i.$$

Eq.(3.39) can be obtained from Eq.(2.24) in the limit of $\lambda_0 \rightarrow \infty$ in Eq.(2.25). (Again, the situation for finite λ_0 in Eq.(2.25) does not appear in the NH limit.) Now, the geodesic deviation becomes

$$\frac{d^2 \zeta^i}{d\lambda^2} = 0. \quad (3.40)$$

IV. ON UNIQUENESS OF THE NEWTON-HOOKE LIMIT, PRINCIPLES OF GALILEI-HOOKE'S RELATIVITY AND POSTULATE ON NEWTON-HOOKE UNIVERSAL TIME

A. On Uniqueness of the Newton-Hooke Limit

Usually, the NH limit is defined by both c and $R \rightarrow \infty$, but c/R keeping fixed [1]. It should be noted, however, that such a definition of limit is not well defined because this contraction depends on the realization of the dS group acting on the dS spacetime. Even for the same kinds of coordinates, it also depends on whether the spatial coordinates are dimensional or dimensionless.

The followings are several possible NH limits of de Sitter spacetime.

It has been shown that the BdS spacetime, based on the definition of proper-time simultaneity, has the form in (2.18). Under the NH limit, the geometry becomes

$$d\tau^2 = dT^2, \quad (4.1)$$

and

$$dl^2 = \cosh^2(\nu T) d\tilde{l}_{\Sigma_T}^2. \quad (4.2)$$

In this case, time spans a 1-d Euclidean metric space while the spatial space is conformal to the 3-d Euclidean metric space. The conformal factor depends on time T . As was noted previously, in \mathcal{B}_Λ spacetime we can define two different kinds of simultaneity. In the NH limit, however, the two kinds of definition of simultaneity coincide as was mentioned in the previous section because of Eq.(3.14).

If one introduces dimensionless spatial Beltrami coordinates such that

$$\tilde{x}^i = \frac{1}{R} x^i, \quad (4.3)$$

then

$$\sigma_\Sigma = 1 - x^i x^i / R^2 = 1 - \tilde{x}^i \tilde{x}^i \quad (4.4)$$

and the Robertson-Walker-like metric can be rewritten as

$$ds^2 = c^2 dT^2 - R^2 \cosh^2(cT/R) d\tilde{l}^2, \quad (4.5)$$

where $d\tilde{l}^2$ is the line-element on the unit 3-sphere, which is dimensionless. Now, taking the NH limit, in which the dimensionless coordinates \tilde{x}^i are kept finite, we get

$$d\tau^2 = dT^2 - \frac{1}{\nu^2} \cosh^2(\nu T) d\tilde{l}^2. \quad (4.6)$$

In this case, the concept of spacetime and de Sitter group as well remain even in the NH limit! Namely, the 'contraction' is trivial in the group, algebra and geometry aspects! It is also the case for the Beltrami metric (2.6) if the spatial coordinates chosen as of dimensionless.

Similar uniqueness problem also appears in other forms of de Sitter spacetime. For example, for $k = 0$ de Sitter metric,

$$\begin{aligned} ds^2 &= c^2 d\bar{t}^2 - e^{2c\bar{t}/R} (d\bar{r}^2 + \bar{r}^2 d\Omega^2) \\ \implies \begin{cases} d\tau^2 = d\bar{t}^2, & dl^2 = e^{2\nu\bar{t}} (d\bar{r}^2 + \bar{r}^2 d\Omega^2), & \bar{r} \text{ keeps finite.} \\ d\tau^2 = d\bar{t}^2 - \frac{1}{\nu^2} e^{2\nu\bar{t}} (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2), & \tilde{r} = \bar{r}/R \text{ keeps finite.} \end{cases} \end{aligned} \quad (4.7)$$

For the static de Sitter metric,

$$\begin{aligned}
ds^2 &= (1 - r'^2/R^2)c^2 dt'^2 - \frac{dr'^2}{1 - r'^2/R^2} - r'^2 d\Omega^2 \\
\Rightarrow \begin{cases} d\tau^2 = dt'^2, & dl^2 = dr'^2 + r'^2 d\Omega^2, & r' \text{ keeps finite.} \\ d\tau^2 = (1 - \tilde{r}'^2)dt'^2 - \frac{1}{\nu^2} \left(\frac{d\tilde{r}'^2}{1 - \tilde{r}'^2} + \tilde{r}'^2 d\Omega^2 \right), & \tilde{r}' = r'/R \text{ keeps finite.} \end{cases} \quad (4.8)
\end{aligned}$$

If one requires $\tilde{t} = t/R$, and x keep finite or requires one or two of spatial coordinates $\tilde{x}^i = x^i/R$ keep finite, then one may get different contractions of the same geometry (See, appendix B).

It should be noted that the starting point of Ref.[5] is the static de Sitter metric. Under the NH limit in the first manner in Eq.(4.8), t' is equal to the proper time and the spatial coordinates q^i satisfy $q^i q^i = r'^2$. When such a coordinate system is chosen, free particles in the NH limit of dS spacetime obeys

$$\frac{d^2 q^i}{d\tau^2} = \nu^2 q^i. \quad (4.9)$$

It can be proved that it is the contraction of geodesic equation in the static dS spacetime, too [5]. From Eq.(4.9), one cannot read out that free particles move along straight lines at constant velocities. The relation between coordinates used in [5] and the Beltrami coordinates is given by the second expression of Eq.(3.22) or

$$\tau = \frac{1}{\nu} \tanh^{-1} \nu t,$$

and

$$q^i = \frac{x^i}{\sqrt{1 - \nu^2 t^2}}. \quad (4.10)$$

It should be noted that in terms of the Beltrami coordinates (t, x^i) Eq.(4.9) turns to be Eq.(3.27) of the inertial-type motion and the non-relativistic energy of a particle, Eq.(5) in Ref. [5], turns to be Eq.(3.34).

These examples show that the condition Eq.(3.1) is not enough to determine the limit. Is it possible that there exist one or more physical principles to determine the NH limit uniquely? The answer is yes. In the following subsections, we propose two principles, Galilei-Hooke's relativity principle and postulate on Newton-Hooke universal time t , $|t| \leq \nu^{-1} = R/c$, and show that these two principles should fix the NH limit procedure.

B. Principle of Galilei-Hooke's Relativity

Recall that in Newtonian mechanics, the Galilean relativity principle is respected. The principle states that the laws of mechanics are the same in all inertial frames. In particular, the first law of Newtonian mechanics, which says that a body at rest or in uniform-velocity motion along a straight line remains its state as long as no force acts upon it, is the same in all inertial frames. Therefore, an inertial observer cannot determine by local mechanic experiments whether he is at rest or in a uniform-velocity motion along a straight line.

In the previous discussion on the NH limit, the relativity principle is not considered even though the inertial-type motion has been introduced in both BdS spacetime and NH space-time. We have

seen that if in \mathcal{NH}_+ and the constant curvature spacetime resulted from the trivial contraction in previous subsection, the inertial-type motion, the motion at uniform-velocity along straight line in Sec. IIIB, is regarded as a genuine inertial motion, if a Beltrami coordinate system is regarded as a true inertial coordinate system, and if a set of particles static in a Beltrami coordinate system is regarded as an inertial reference frame, then the inertial motion is independent of the choice of inertial frames. The statement may serve as the counterpart of the first law of Newtonian mechanics in \mathcal{NH}_+ and can be expressed in the same way as the first Newton's law of mechanics: in \mathcal{NH}_+ a body at rest or in uniform-velocity motion along a straight line remains its state as long as no force acts upon it. The counterpart of the second Newton's law of mechanics may also be set up in \mathcal{NH}_+ as

$$\frac{dP^i}{dt} = F^i. \quad (4.11)$$

In \mathcal{NH}_+ , both sides of Eq.(4.11) should transform in the same manner under the transformation Eq.(3.4). Now, we may introduce a relativity principle parallel to the Galilean relativity principle in Newtonian mechanics as follows:

The laws of mechanics in the Newton-Hooke space-time are the same in all inertial frames.

We name it Galilei-Hooke's relativity principle in honor of Galilei and Hooke. (From now on, the '-type' in phrases, such as, inertial-type motion, inertial-type observer, inertial-type frame, are removed.)

By use of Galilei-Hooke's relativity principle, the contraction like the third example in Appendix B can be excluded. Furthermore, the limit that $c \rightarrow \infty$ while R keeps finite can also be excluded by the Galilei-Hooke's relativity principle. Therefore, the requirement $R \rightarrow \infty$ in the non-relativistic approximation of special-like relativity on \mathcal{B}_Λ [6] can be relaxed because it is the result of the Galilei-Hooke's relativity principle.

C. Postulate on Newton-Hooke Universal Time

It should be noted that even when the Galilei-Hooke's relativity principle is respected, it is still not enough to fix the NH limit uniquely.

In order to fix the contraction procedure, it is needed to explore what should happen at any given point as the origin of a coordinate system chosen. In the case of BdS , the metric at origin is Minkowskian and under the NH contraction, it splits to two parts. The similar situation also happens in the case of static dS metric. Since both spacetime \mathcal{B}_Λ and space-time \mathcal{NH}_+ are homogeneous and isotropic, all points should follow this behavior. Note that for the metric at the origin, there is no room for R . Thus, this may fix the contraction procedure. This may also be seen from both the light-cone and the homogeneous Lorentz group at the origin. In fact, we should define the inertial motion, inertial coordinates as well as inertial observers and require two first principles for BdS : the relativity principle and the postulate on universal constants, which require that there exist two universal constants of c with dimension of velocity and R with dimension of length. As was indicated in [6], the second postulate implies that the light cone at the origin is the same as that in a Minkowski spacetime. The lack of the two principles seems to indicate in mathematics that the contraction procedure may not be fixed by only considering algebraic contraction.

Therefore, we propose the second physical principle, the postulate on the Newton-Hooke universal time:

There exists a bounded Newton-Hooke universal time $t, |t| < \nu^{-1} = R/c$ under the NH limit of $c, R \rightarrow \infty$, which measures the time of physical processes.

Thus, under the postulate, especially at origin of the (Beltrami) coordinates, the metric splits to two parts under the limit of $c \rightarrow \infty$. In brief, only considering algebraic contraction is not enough to uniquely determine the NH limit. One has to supplement two physical principles. One is the Galilei-Hooke's relativity principle and the other is the postulate on Newton-Hooke universal time.

V. FURTHER STUDY ON INERTIAL REFERENCE FRAMES

A. Relative Velocity, Inertial Frame and Observers

In Newtonian mechanics and special relativity, if two particles have the same velocity relative to one inertial frame, they have the same velocity relative to every inertial frame. As a consequence, there cannot be a particle that is at rest at the same time in two different inertial frames. In addition, in Newtonian mechanics and special relativity, any two inertial frames S and S' have the property that all particles being at rest in S' have the same velocity relative to S . In the following, we shall see these are not the facts in \mathcal{NH}_+ .

Similar to what in Newtonian mechanics and special relativity, the equation of motion for a free particle is Eq.(3.26). Especially, when $t_a = 0$, two coordinate systems for S and S' related by the transformation (3.4) have the same relation as that in Newtonian mechanics, except the coordinate time t taking value in $(-\nu^{-1}, \nu^{-1})$ in \mathcal{NH}_+ . When $t_a \neq 0$, however, the velocities of free particles at rest in S' are different from one another in S . This can be seen from the equation of motion of a static particle in S' in terms of S ,

$$x^i = a^i + \sigma_+^{-1/2}(t_a) O_j^i x'^j + [u^i - \nu^2 t_a \sigma_+^{-1/2}(t_a) O_j^i x'^j] t. \quad (5.1)$$

Obviously, when $t_a \neq 0$, there is a special fixed spatial point

$$x^i = \frac{\sigma_+^{1/2}(t_a)}{\nu^2 t_a} O_j^i u^j \quad (5.2)$$

in S' , whose velocity relative to S is also zero! Namely, its spatial coordinates in S is also fixed at

$$x^i = a^i + \frac{u^i}{\nu^2 t_a}. \quad (5.3)$$

Therefore, when $t_a \neq 0$, there is always a free particle P_0 at rest in both frames S and S' . Let us select another two distinct particles P and P' such that P is static in S while P' is static in S' . From the point of view of an observer in S , P_0 and P have the same velocity while P_0 and P' have different velocities. From the point of view of an observer in S' , it is on the contrary: P_0 and P' have the same velocity while P_0 and P have different velocities. So, in this case, the observation that two particles have the same velocity is a phenomenon depending on reference frames: Two frames could share a common particle that is static in both of them. For such a particle it could say that another remote particle is static relative to it while, at the same time, it could deny this statement. The answer depends on which reference frame it thinks itself belongs to.

Customarily, we use an inertial observer to take the place of an inertial frame in which he is static. This is due to the properties of inertial frames in Newtonian mechanics and special relativity. From the above discussion we find that this is misleading in \mathcal{NH}_+ , because the same inertial observer can be static in distinct inertial frames. In fact, as we can see more clearly in next section, the choice of the time origin of NH inertial coordinate systems has much more nontrivial meaning than the Newtonian case. Especially, the concept of static state is dependent on this choice.

At last, we point out that, whenever $t_a \neq 0$, we can shift the spatial origins of S and S' , respectively, so that the transformation from S to S' have the standard form as

$$t' = \frac{t - t_a}{\sigma_+(t_a, t)}, \quad x'^i = \frac{\sigma_+^{1/2}(t_a)}{\sigma_+(t_a, t)} O^i_j x^j. \quad (5.4)$$

B. On Euclid's Fifth Axiom

In this subsection we try to give a clear geometric picture to the NH transformations. We can see the shadow of projective geometry.

In \mathcal{NH}_+ a 2-d surface

$$t = u^0, \quad x^i = V^i u^0 + A^i u^1 + a^i \quad (\mathbf{V}^2 + \mathbf{A}^2 \neq 0) \quad (5.5)$$

is called a timelike plane and a 2-d surface

$$t = \text{const}, \quad x^i = A_0^i u^0 + A_1^i u^1 + a^i \quad (\mathbf{A}_0 \times \mathbf{A}_1 \neq 0) \quad (5.6)$$

is called a spacelike plane, where u^0 and u^1 are parameters describing the surfaces and V^i , A^i 's and a^i are some constants. Since Eqs.(5.5) and (5.6) remain the same forms, respectively, under the NH transformations, the above concepts are well defined.

It can be shown that straight lines on the planes in \mathcal{NH}_+ , which can be expressed by

$$k_0 u^0 + k_1 u^1 = l \quad (5.7)$$

in terms of the parameters u^0 and u^1 , are also straight lines (i.e. geodesics) in \mathcal{NH}_+ . In Eq.(5.7), k_0 , k_1 and l are some constants, and u^0 takes value in $(-\nu^{-1}, \nu^{-1})$ for timelike planes and $(-\infty, \infty)$ for spacelike planes. Two geodesics in a plane are called parallel lines if they have no common points. It is obviously that Euclid's fifth axiom is valid on spacelike planes but invalid on timelike planes. Applying the same ideas to discuss spacetimes of Newtonian mechanics or special relativity, we can find that Euclid's fifth axiom is valid for every plane. This is one of the main difference between \mathcal{NH}_+ and the spacetime in Newtonian mechanics or the special relativity.

The concept of straight line, plane and parallel are independent of the choice of Beltrami coordinate systems. That is, if a curve is a straight line in one coordinate system, it is a straight line in all coordinate systems; If two straight lines are parallel in one coordinate system, they are parallel in all coordinate systems. For an inertial frame S , the world lines $\mathbf{x} = \mathbf{x}_1$ and $\mathbf{x} = \mathbf{x}_2$ are straight lines that are parallel to each other. From the point of view other inertial frame, S' , say, these world lines are still parallel straight lines. It does represent the invalidity of Euclid's fifth axiom that there exists an observer who can sit statically in both S and S' .

To be clearer, we give some diagrams to the standard transformation (5.4). Define

$$t_* = \frac{1}{\nu^2 t_a}. \quad (5.8)$$

It is meaningful for all nonzero t_a and can be generalized to the case $t_a = 0$, that can be regarded as a special case in which t_* is at infinity on the time axis. It should be noted that t_* is not a possible coordinate time because $|\nu t_*| > 1$. For simplicity we only consider a plane Σ containing the world line of the spatial origin of the systems, namely, $\mathbf{x} = 0$ and $\mathbf{x}' = 0$. If the world lines of $x = \text{const}$ are shown as in Fig.1, the world lines of $x' = \text{const}$ will be shown, in the same coordinate system, as in Fig.2. In this figure, these world lines are parallel to each other and are focused on the point $t = t_*$

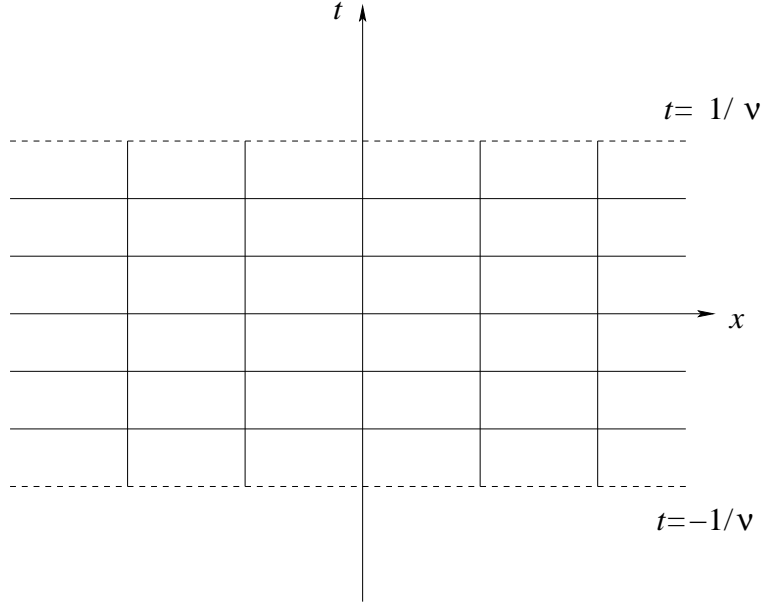


FIG. 1: World lines of fixed points $x = \text{const}$. Horizontal lines are $t = \text{const}$.

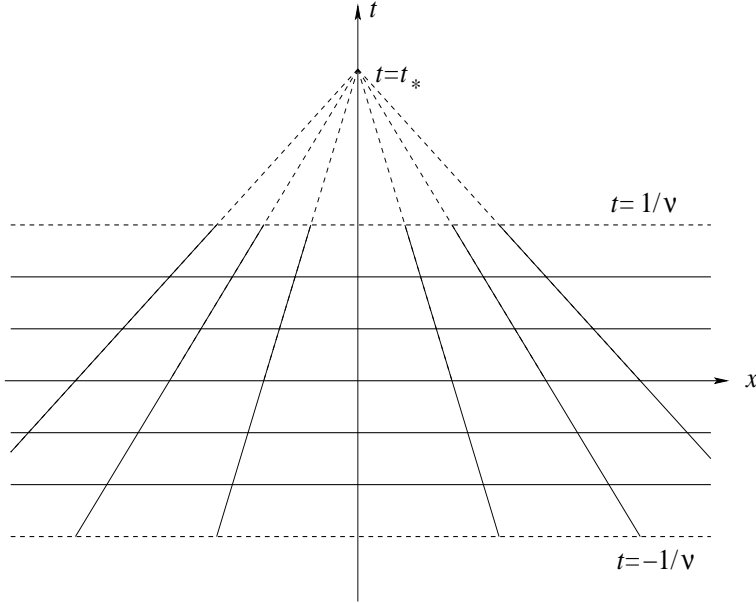


FIG. 2: World lines of fixed points $x' = \text{const}$. Horizontal lines are $t = \text{const}$.

on the time axis. Conversely if a congruence of geodesics is focused on a point in $|t| > \nu^{-1}$ (not necessarily on the time axis), we can say that they are the world lines of rest particles in certain an inertial frame. Due to the nature of fractional linear transformations, the cases in Fig.1 and Fig.2 are symmetric. If we draw the world lines in the x' - t' coordinate system, then the world lines $x' = \text{const}$ will look like those in Fig.1, while the world lines $x = \text{const}$ will be focused on $t' = -t_*$.

Fig.3 is a graphic illustration for the transformation of coordinate time. In the diagram, the horizontal t -axis and the vertical t' -axis form a Cartesian coordinate system. The point C is located at $(t_*, -t_*)$ where t_* is given in Eq.(5.8). Choose points T and T' such that they are collinear with C . Let the coordinates of T and T' be $(t, -1/\nu)$ and $(-1/\nu, t')$, respectively. Then, t and t' can be linked by the transformation (5.4). From Eq.(5.4) and the diagram, we can see $\pm 1/\nu$ are invariant

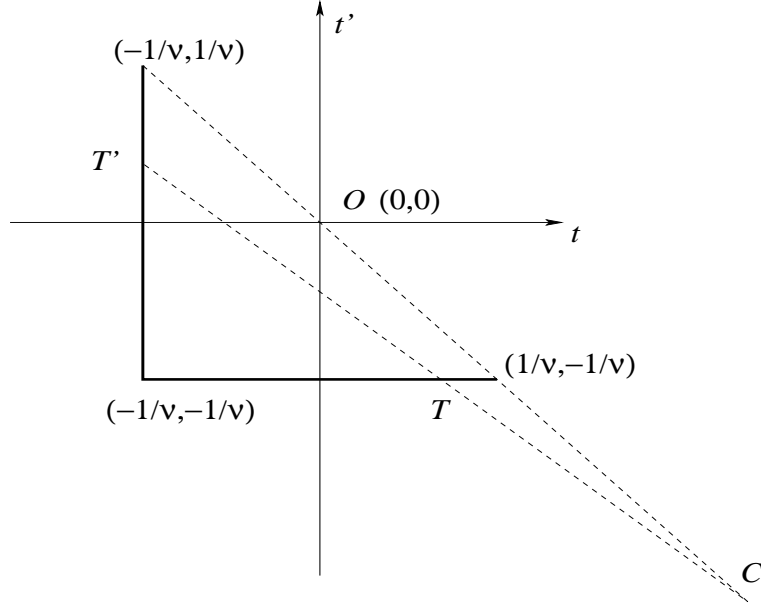


FIG. 3: Diagram for transformation of coordinate time.

time in \mathcal{NH}_+ . Compared with special relativity in which there exists an invariant velocity c , the NH mechanics is a kind of relativity due to the existence of finite extremum time.

VI. CONCLUDING REMARKS

We have shown that in \mathcal{B}_Λ , the transformation group between Beltrami coordinate systems contracts to NH_+ group in the NH limit. The geometry of BdS spacetime splits time and spatial space in the NH limit. The former may be regarded as a subspace of \mathbb{RP}^1 with metric², while the latter is conformal to a 3-d Euclidean space. The conformal factor depends on time only. Test particles and light signals without force acting upon them move along straight lines at uniform velocity in the NH limit, as expected. The behavior of uniform-velocity motion along straight lines is independent of the inertial frame which observer is in and the velocity addition law is obtained. This can be regarded as the key point of Galilei-Hooke's relativity principle, which together with the postulate on Newton-Hooke universal time determines the NH limit uniquely.

The contraction of BdS spacetime shows that the Beltrami coordinate systems should be regarded as inertial coordinate system both in relativistic and non-relativistic levels since test particles and signals without force acting upon them all move along straight lines at uniform velocities. This implies that the mechanics in non-zero constant-curvature space-time may be set up in parallel to the one in flat space-time, starting from the relativity principle and inertial law. The dynamics both at classical and quantum level will be studied elsewhere.

It is remarkable that unlike in Newtonian mechanics and special relativity, a free particle in \mathcal{NH}_+ (and \mathcal{B}_Λ) can be static at the same time with respect to two different inertial frames which have relative velocity. This property is closely related to the violation of Euclid's fifth axiom. In this sense, the NH limit of BdS spacetime can be regarded as the version of the Newtonian mechanics without Euclid's fifth axiom.

² The cross ratio is an invariant and the metric is its infinitesimal form.

Another remarkable property of \mathcal{NH}_+ is that there is a universal time. It is similar to the Newton's universal time in Newtonian mechanics in the sense that it is separated from space in the metric and that the simultaneity is absolute. However, in any inertial coordinate system, this universal time as the coordinate time should be in the range $(-1/\nu, 1/\nu)$. Compared with special relativity in which there exists an invariant velocity c , the NH mechanics looks like a kind of relativity due to the existence of finite extremum universal time. It is related to the fact that the generators of spatial translation and boost along the same direction form an $SO(1, 1)$ vector under the time translation. [See, Eqs.(3.7), (3.10), and (3.12).] Further studies on space-time, group structures, and kinematics and dynamics in the NH limit are needed.

It should be noted that when the NH limit in \mathcal{B}_Λ is taken, the zeroth coordinate $\xi^0 \rightarrow \infty$. In order Eq.(2.1) to be meaningful, one of ξ^a ($a = 1, 2, 3, 4$), at least, should tend to ∞ . For example, in U_4 , $\xi^4 \rightarrow \infty$ while ξ^i ($i = 1, 2, 3$) keep finite so that $x^i < \infty$ from Eq.(2.3). The intersection of U_3 and U_4 , $U_4 \cap U_3$, can only appears at ξ^3, ξ^4 (and thus x^3, x^4) $\rightarrow \infty$.

When $\nu = 0$, namely, $R \rightarrow \infty$ faster than $c \rightarrow \infty$, all results become the correspondences in flat space-time. In particular, the Galilei-Hooke's relativity principle reduces to the Galilean one. Non-relativistic energy of a test particle reduces to the one in the Newtonian mechanics. The NH group reduces to the Galilei group. The Newton-Hooke universal time becomes Newton's universal time.

In the present paper, we only discuss the contraction of BdS spacetime which has a positive cosmological constant. All results are readily extended to the contraction of Beltrami-anti-de Sitter spacetime which has negative cosmological constant.

APPENDIX A: ON CONNECTION AND CURVATURE

As pointed out in Sec. II A and in [6–8], in the Beltrami-de Sitter spacetime \mathcal{B}_Λ the metric

$$g_{\mu\nu} = \eta_{\mu\nu}\sigma^{-1}(x) + R^{-2}\eta_{\mu\rho}\eta_{\nu\sigma}x^\rho x^\sigma \sigma^{-2}(x)$$

and its inverse

$$g^{\mu\nu} = \sigma(x)(\eta^{\mu\nu} - R^{-2}x^\mu x^\nu) \quad (\text{A1})$$

are invariant under the fractional linear transformation (2.7). Consequently, the connection coefficients Eq.(2.21) and the components of the curvature tensor

$$R^\mu_{\nu\rho\sigma} = R^{-2}(g_{\nu\rho}\delta^\mu_\sigma - g_{\nu\sigma}\delta^\mu_\rho), \quad (\text{A2})$$

have the same property. Since the coordinate transformation from one patch to another is a special case of the fractional linear transformation (2.7), the forms of these quantities are the same in all the coordinate patches and the expressions of the geodesics have the same form in all the patches, too.

The connection on \mathcal{B}_Λ is torsion free. The connection on \mathcal{NH}_+ is torsion free, too. Since not a particular Beltrami coordinate system is specified in obtaining Eq.(3.19) from the contraction of coefficients of connection, Eq.(3.19) should remain the same form under the NH transformations. In fact, we can define a new set of symbols $\Gamma'^\rho_{\mu\nu}$ among which only the following are nonzero:

$$\Gamma'^t_{tt} = \frac{2\nu^2 t'}{1 - \nu^2 t'^2}, \quad \Gamma'^i_{tj} = \Gamma'^i_{jt} = \frac{\nu^2 t'}{1 - \nu^2 t'^2} \delta^j_i.$$

Then it can be verified that $\Gamma^\rho_{\mu\nu}$ and $\Gamma'^\rho_{\mu\nu}$ satisfy the standard transformation relation for coefficients of connection if $(t, \mathbf{x}) \rightarrow (t', \mathbf{x}')$ is an NH transformation. So, the connection in \mathcal{NH}_+ is well defined.

The nonzero components of the curvature tensor and Ricci tensor calculated from Eq.(3.19) are, respectively

$$R_{t\mu\nu}^i = \frac{\nu^2}{(1 - \nu^2 t^2)^2} (\delta_\mu^t \delta_\nu^i - \delta_\mu^i \delta_\nu^t) \quad \text{and} \quad R_{tt} = -\frac{3\nu^2}{(1 - \nu^2 t^2)^2}. \quad (\text{A3})$$

It follows that the mathematical forms of the curvature tensor and Ricci tensor are also invariant under the NH transformation. And, on the other hand, they can also be obtained by contracting the curvature tensor and Ricci tensor on \mathcal{B}_Λ , respectively.

Finally, we re-study the deviation equation of geodesics in \mathcal{NH}_+ in terms of 4-d language.

Given a timelike geodesic γ in \mathcal{NH}_+ , we can construct a family of geodesics, $t = \nu^{-1} \tanh^{-1} \nu \tau$, $x^i = x^i(\tau, u)$, labeled by the parameter u . For convenience, it will be denoted by $x^\mu = x^\mu(\tau, u)$ with $\mu = t, 1, 2$ and 3 and $x^t := t$. Assume that the geodesic labeled by $u = 0$ is γ . Then, for each geodesic, Eqs.(3.20) and (3.21) are equivalent to

$$\frac{\partial^2 x^\mu}{\partial \tau^2} + \Gamma_{\rho\sigma}^\mu \frac{\partial x^\rho}{\partial \tau} \frac{\partial x^\sigma}{\partial \tau} = 0, \quad (\text{A4})$$

where $\frac{\partial x}{\partial \tau} := \frac{\partial x(\tau, u)}{\partial \tau}$, etc. The deviation ζ^μ is a vector field along γ , defined by

$$\zeta^\mu(\tau) = \left. \frac{\partial x^\mu}{\partial u} \right|_{u=0}. \quad (\text{A5})$$

Obviously, $\zeta^t = 0$. If we take the partial derivative with respect to u on both sides of Eq.(A4) then set $u = 0$, we can obtain the derivation equation

$$\frac{D^2 \zeta^\mu}{d\tau^2} + R_{\nu\rho\sigma}^\mu \dot{\gamma}^\nu \zeta^\rho \dot{\gamma}^\sigma = 0, \quad (\text{A6})$$

where $\dot{\gamma}^\nu$ is the tangent vector of γ .

Using the expression of connection coefficients and the fact that $\zeta^t = 0$, we can verify that the t -component of the covariant derivative of ζ^μ is always zero. It follows that, when $\mu = t$, Eq.(A6) is an identity. When $\mu = i$, it is equivalent to

$$\frac{D^2 \zeta^i}{d\tau^2} = \nu^2 \zeta^i, \quad \text{namely,} \quad \frac{d^2 \zeta^i}{dt^2} = 0. \quad (\text{A7})$$

APPENDIX B: GEOMETRIC POINT OF VIEW OF THE NEWTON-HOOKE LIMIT

We have seen that the result of the limit (3.1) is not unique. In addition to Eq.(3.1), we need to specify the limits of the coordinates. We should have also noticed that the topological structure of contraction may not be the same as that of \mathcal{B}_Λ . Here, we try to give a geometric point of view of the contraction.

First, for a given positive $\Lambda = 3R^{-2}$, \mathcal{B}_Λ is viewed as a hyperboloid of the 5-d Minkowski spacetime $\mathcal{M}^{1,4}$, as shown in section II A. When Λ , or R , runs over all the positive values, a family of Beltrami-de Sitter spacetimes are obtained. Obviously, through each spacelike point in $\mathcal{M}^{1,4}$, there is one and only one BdS spacetime in the family.

Roughly speaking, to specify a particular limit evolving $R \rightarrow \infty$, one must specify a region $D \subset \mathcal{M}^{1,4}$ consisting of spacelike points of $\mathcal{M}^{1,4}$ and a congruence of curves in D satisfying the following conditions.

(1) Through each spacelike point in D there is one and only one curve in the congruence.

(2) At each spacelike point in D , the corresponding curve in the congruence is not tangent to \mathcal{B}_Λ passing through the same point. Consequently, there is a direction along each of these curves such that the parameter R increases strictly. That is, R can be viewed as the parameter for each curve in the congruence.

(3) On each curve, the parameter R can take any sufficiently large positive value.

(4) The congruence is smooth in the following sense. The congruence of curves can be described using four parameters t, x^i ($i = 1, 2, 3$), say, such that t, x^i and R is a set of local coordinates on D when R is sufficiently large.

Given such a region D and a congruence \mathcal{C}_D of curves in D , which will be denoted by (D, \mathcal{C}_D) , these curves can be viewed as the orbits of an associated limit process. Thus, on a given curve γ in the congruence \mathcal{C}_D , points with different values of the parameter R are viewed as the “same” point in the limit process. Each of these curves can be identified with the limit point of the process. And the parameters t, x^i can be regarded as the coordinates of the limit point.

So, different region or different congruence of curves give different limit processes. In the following we will show some examples.

The first example is the NH limit obtained in section III A. The region D is the set of spacelike points ξ^A in $\mathcal{M}^{1,4}$ such that $\xi^4 > 0$. Obviously, D is homeomorphic to \mathbb{R}^5 . The congruence $\mathcal{C}_{D,\nu}$ consists of curves $\xi^A = \xi^A(R; t, x^1, x^2, x^3; \nu)$ with

$$\{\xi^0, \xi^i, \xi^4\} = (1 - \nu^2 t^2 + R^{-2} |\mathbf{x}|^2)^{-1/2} \{\nu t R, x^i, R\}, \quad (\text{B1})$$

where R is the parameter on the curve, t and \mathbf{x} are parameters describing curves in $\mathcal{C}_{D,\nu}$, and ν is a parameter to distinguish one congruence from another. Equivalently, the limit process associated to the above congruence is to keep $t = \xi^0/(\nu \xi^4)$ and $x^i = R \xi^i/\xi^4$ finite as R tends to infinity. At the same time, $c = \nu R$ also tends to infinity.

For the second example, let the region D be the set of all spacelike points in $\mathcal{M}^{1,4}$ and $\xi^A = \xi^A(R; T, \tilde{\mathbf{x}}; \nu)$ with

$$\{\xi^0, \xi^i, \xi^4\} = (1 - \nu^2 T^2 + |\tilde{\mathbf{x}}|^2)^{-1/2} \{\nu T R, R \tilde{x}^i, R\} \quad (\text{B2})$$

then the congruence consisting of curves $\xi^A = \xi^A(R; T, \tilde{\mathbf{x}}; \nu)$ corresponds to the limit process as shown in section IV A. In this limit process, $c = \nu R$ tends to infinity while T and the dimensionless spatial coordinates $\tilde{\mathbf{x}}$ are kept finite. Obviously this also meets the requirement of NH limit. The resulted spacetime is, once again, a de Sitter spacetime. See, section IV A.

Now let us consider the last example. The region D will be the set of spacelike points in $\mathcal{M}^{1,4}$ such that $\xi^3 > 0$. In the limit process, both R and c tend to infinity with $\nu = c/R$ kept fixed. The region $D = D_{+1} \cup D_{-1} \cup D_{+2} \cup D_{-2} \cup D_{+4} \cup D_{-4}$ where $D_{\pm 4}$, for example, is the subset of D on which ξ^4 is positive/negative. Beltrami coordinates can be defined on each of these subsets. For example, on $D_{\pm 4}$, the Beltrami coordinates can be defined to be

$$t = \frac{\xi^0}{\nu \xi^4}, \quad x^a = \frac{\xi^a}{\xi^4}, \quad (a = 1, 2), \quad z = R \frac{\xi^3}{\xi^4}. \quad (\text{B3})$$

Note that νt , x^1 and x^2 are dimensionless while z has a dimension of length.

We assume that all the coordinates are kept finite in the limit process. In other words, we are considering a congruence of curves³ $\xi^A = \xi^A(R; t, x^1, x^2, z; \nu)$ on D with

$$\{\xi^0, \xi^1, \xi^2, \xi^3, \xi^4\} = \tilde{\sigma}^{-1/2}(t, x^1, x^2, z) \{\nu t R, x^1 R, x^2 R, z, R\}, \quad (\text{B4})$$

³ In fact, the parameters t, x^a and z could only describe part of the curves in the congruence. These parameters corresponds to the Beltrami coordinates on $D_{\pm 4}$. There exist some curves in the congruence that must described by parameters corresponding to the coordinates on $D_{\pm 1}, D_{\pm 2}$, respectively.

where

$$\tilde{\sigma}(t, x^1, x^2, z) = 1 - \nu^2 t^2 + (x^1)^2 + (x^2)^2 + \left(\frac{z}{R}\right)^2. \quad (\text{B5})$$

The parameters t , x^a , z and R can be regarded to be coordinates on $D_{\pm 4} \subset D$. The 5-d Minkowski metric is then

$$ds^2 = \frac{R^2}{\tilde{\sigma}} \left(\nu^2 dt^2 - (dx^1)^2 - (dx^2)^2 - \frac{1}{R^2} dz^2 \right) + R^2 \left(\sqrt{\tilde{\sigma}} d\frac{1}{\sqrt{\tilde{\sigma}}} \right)^2 - \left(1 + \frac{z^2}{R^2 \tilde{\sigma}} \right) dR^2 + \frac{2z}{R \tilde{\sigma}} dz dR, \quad (\text{B6})$$

where

$$\sqrt{\tilde{\sigma}} d\frac{1}{\sqrt{\tilde{\sigma}}} = \frac{1}{\tilde{\sigma}} \left(\nu^2 t dt - x^1 d(x^1) - x^2 d(x^2) - \frac{z}{R^2} dz + \frac{z^2}{R^3} dR \right).$$

To obtain the limit, we first consider the induced metric on a hypersurface $R = \text{const}$, on which $dR = 0$. Then we take the limit of $c^{-2} ds^2$ as $R \rightarrow \infty$. Obviously, its limit is

$$d\tau^2 = \nu^{-2} g_{\alpha\beta} dx^\alpha dx^\beta, \quad (\text{B7})$$

where α and β take values from 0 to 2, with $x^0 = \nu t$ and

$$g_{\alpha\beta} = \frac{\eta_{\alpha\beta}}{\sigma_{BdS_3}(x)} + \frac{\eta_{\alpha\alpha'} \eta_{\beta\beta'} x^{\alpha'} x^{\beta'}}{\sigma_{BdS_3}^2(x)}, \quad \sigma_{BdS_3}(x) = 1 - \eta_{\alpha\beta} x^\alpha x^\beta. \quad (\text{B8})$$

Needless of speaking, we obtain a space-time having the structure of $BdS_3 \times \mathbb{R}$, where BdS_3 is a unit 3-d Beltrami-de Sitter spacetime, while \mathbb{R} is labeled by the coordinate z .

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